

The Field Nature of Time in General Relativity

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Abstract

The paper is devoted to the description of the reparametrization - invariant dynamics of general relativity obtained by resolving constraints and constructing equivalent unconstrained systems. The constraint-shell action reveals the "field nature" of the geometric time in general relativity. The time measured by the watch of an observer coincides with one of field variables, but not with the reparametrization-noninvariant coordinate evolution parameter.

We give new solution of such problem, as the derivation of the path integral representation of the causal Green functions in the Hamiltonian scheme of general relativity.

1. Introduction

The problem of time and the reparametrization - invariant Hamiltonian description of general relativity has a long history [1] - [7]. There are two opposite approaches to solution of this problem in the generalized Hamiltonian formulation [1, 8, 9, 10, 11, 12].

The first approach is the reduction of the extended phase space by fixing the gauge that breaks reparametrization invariance from the very beginning [2, 6].

The second approach is the reparametrization-invariant reduction of an action by the explicit resolving of the first class constraints to get an equivalent unconstrained system, so that one of the variables of the extended phase space (with a negative contribution to the energy constraint) converts into the *dynamic evolution parameter*, and its conjugate momentum becomes the nonzero Hamiltonian of evolution [3, 4, 7, 13, 14, 15, 16].

An example of the application of such an invariant reduction of the action is the Dirac formulation of QED [17] directly in terms of the gauge-invariant (dressed) fields as the proof of the adequateness of the Coulomb gauge with the invariant content of classical equations. As it was shown by Faddeev [18], the invariant reduction of the action is the way to obtain the unconstrained Feynman integral for the foundation of the intuitive Faddeev-Popov functional integral in the non-Abelian gauges theories [19, 20].

The application of the invariant reduction of extended actions in cosmology and general relativity [4, 7] allows one to formulate the dynamics of relativistic systems directly in terms of the invariant geometric time with the nonzero Hamiltonian of evolution, instead of the non-invariant coordinate time with the generalized zero Hamiltonian of evolution in the gauge-fixing method. The formulation in terms of the geometric time is based on the Levi-Civita canonical transformation [12, 21] that converts the energy constraint into a new momentum, so that the new dynamic evolution parameter coincides with the geometric time, as one of the consequences of new equations of motion.

In the present paper, we apply the method of the invariant Hamiltonian reduction (with resolving the first class constraints) to express reparametrization-invariant dynamics of relativistic systems in terms of the geometric time and to construct the causal Green functions in the form of the path integrals in the world space of dynamic variables.

2. Hamiltonian Dynamics of General Relativity

2.1. Action and geometry

General relativity (GR) is given by the singular Einstein-Hilbert action with the matter fields

$$W(g|\mu) = \int d^4x \sqrt{-g} \left[-\frac{\mu^2}{6} R(g) + \mathcal{L}_{matter} \right] \quad \left(\mu^2 = M_{Planck}^2 \frac{3}{8\pi} \right) \quad (1)$$

and by a measurable interval in the Riemannian geometry

$$(ds)^2 = g_{\alpha\beta} dx^\alpha dx^\beta . \quad (2)$$

They are invariant with respect to general coordinate transformations

$$x_\mu \rightarrow x'_\mu = x'_\mu(x_0, x_1, x_2, x_3). \quad (3)$$

2.2. Variables and Hamiltonian

The generalized Hamiltonian approach to GR was formulated by Dirac and Arnovit, Deser and Misner [2] as a theory of system with constraints in $3+1$ foliated space-time

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu = N^2 dt^2 - {}^{(3)}g_{ij} \check{dx}^i \check{dx}^j \quad (\check{dx}^i = dx^i + N^i dt) \quad (4)$$

with the lapse function $N(t, \vec{x})$, three shift vectors $N^i(t, \vec{x})$, and six space components ${}^{(3)}g_{ij}(t, \vec{x})$ depending on the coordinate time t and the space coordinates \vec{x} . The Dirac-ADM parametrization of metric (4) characterizes a family of hypersurfaces $t = \text{const.}$ with the unit normal vector $\nu^\alpha = (1/N, -N^k/N)$ to a hypersurface and with the second (external) form

$$\frac{1}{N} ({}^{(3)}\dot{g}_{ij}) - \Delta_i N_j - \Delta_j N_i \quad (5)$$

that shows how this hypersurface is embedded into the four-dimensional space-time.

Coordinate transformations conserving the family of hypersurfaces $t = \text{const.}$

$$t \rightarrow \tilde{t} = \tilde{t}(t); \quad x_i \rightarrow \tilde{x}_i = \tilde{x}_i(t, x_1, x_2, x_3) , \quad (6)$$

$$\tilde{N} = N \frac{dt}{d\tilde{t}}; \quad \tilde{N}^k = N^i \frac{\partial \tilde{x}^k}{\partial x_i} \frac{dt}{d\tilde{t}} - \frac{\partial \tilde{x}^k}{\partial x_i} \frac{\partial x^i}{\partial \tilde{t}} \quad (7)$$

are called a kinematic subgroup of the group of general coordinate transformations (3) [22, 4, 5, 7]. The group of kinematic transformations is the group of diffeomorphisms of the generalized Hamiltonian dynamics. It includes reparametrizations of the nonobservable time coordinate $\tilde{t}(t)$ (6) that play the principal role in the procedure of the reparametrization-invariant reduction discussed in the previous Sections. The main assertion of the invariant reduction is the following: the dynamic evolution parameter is not the coordinate but the variable with a negative contribution to the energy constraint. (Recall that this reduction is based on the explicit resolving of the global energy constraint with respect

to the conjugate momentum of the dynamic evolution parameter to convert this momentum into the Hamiltonian of evolution of the reduced system.)

A negative contribution to the energy constraint is given by the space-metric-determinant logarithm. Therefore, following papers [3, 4, 13, 14, 23, 24] we introduce an invariant evolution parameter $\varphi_0(t)$ as the zero Fourier harmonic component of this logarithm (treated, in cosmology, as the cosmic scale factor). This variable is distinguished in general relativity by the Lichnerowicz conformal-type transformation of field variables f with the conformal weight (n)

$${}^{(n)}\bar{f} = {}^{(n)}f \left(\frac{\varphi_0(t)}{\mu} \right)^{-n}, \quad (8)$$

where $n = 2, 0, -3/2, -1$ for the tensor, vector, spinor, and scalar fields, respectively, \bar{f} is so-called conformal-invariant variable used in GR for the analysis of initial data [23, 25]. In particular, for metric we get

$$g_{\mu\nu}(t, \vec{x}) = \left(\frac{\varphi_0(t)}{\mu} \right)^2 \bar{g}_{\mu\nu}(t, \vec{x}) \Rightarrow (ds)^2 = \left(\frac{\varphi_0(t)}{\mu} \right)^2 [\bar{N}^2 dt^2 - {}^{(3)}\bar{g}_{ij} \check{d}x^i \check{d}x^j]. \quad (9)$$

As the zero Fourier harmonic is extracted from the space metric determinant logarithm, the space metric $\bar{g}_{ij}(t, \vec{x})$ should be defined in a class of nonzero harmonics

$$\int d^3x \log ||\bar{g}_{ij}(t, \vec{x})|| = 0. \quad (10)$$

The transformational properties of the curvature $R(g)$ with respect to the transformations (9) lead to the action (1) in the form [4]

$$W(g|\mu) = W(\bar{g}|\varphi_0) - \int_{t_1}^{t_2} dt \int_{V_0} d^3x \varphi_0 \frac{d}{dt} \left(\frac{\dot{\varphi}_0 \sqrt{\bar{g}}}{\bar{N}} \right). \quad (11)$$

This form define the global lapse function N_0 as the average of the lapse function \bar{N} in the metric \bar{g} over the kinematic invariant space volume

$$N_0(t) = \frac{V_0}{\int_{V_0} d^3x \frac{\sqrt{\bar{g}(t, \vec{x})}}{\bar{N}(t, \vec{x})}}, \quad \bar{g} = \det({}^{(3)}\bar{g}), \quad V_0 = \int_{V_0} d^3x, \quad (12)$$

where V_0 is a free parameter which in the perturbation theory has the meaning of a finite volume of the free coordinate space. The lapse function $\bar{N}(t, \vec{x})$ can be factorized into the global component $N_0(t)$ and the local one $\mathcal{N}(t, \vec{x})$

$$\bar{N}(t, \vec{x}) \bar{g}^{-1/2} := N_0(t) \mathcal{N}(t, \vec{x}) := N_q, \quad (13)$$

where \mathcal{N} fulfills normalization condition:

$$I[\mathcal{N}] := \frac{1}{V_0} \int \frac{d^3x}{\mathcal{N}} = 1 \quad (14)$$

that is imposed after the procedure of variation of action, to reproduce equations of motion of the initial theory. In the Dirac harmonical variables [1] chosen as

$$q^{ik} = \bar{g}^{ik}, \quad (15)$$

the metric (4) takes the form

$$(ds)^2 = \frac{\varphi_0(t)^2}{\mu^2} q^{1/2} \left(N_q^2 dt^2 - q_{ij} \check{d}x^i \check{d}x^j \right), \quad (q = \det(q^{ij})). \quad (16)$$

The Dirac-Bergmann version of action (11) in terms of the introduced above variables reads [4, 5]

$$W = \int_{t_1}^{t_2} dt \left\{ L + \frac{1}{2} \partial_t (P_0 \varphi_0) \right\}, \quad (17)$$

$$L = \left[\int_{V_0} d^3x \left(\sum_F P_F \dot{F} - N^i \mathcal{P}_i \right) \right] - P_0 \dot{\varphi}_0 - N_0 \left[-\frac{P_0^2}{4V_0} + I^{-1} H(\varphi_0) \right], \quad (18)$$

where

$$\sum_F P_F \dot{F} = \sum_f p_f \dot{f} - \pi_{ij} \dot{q}^{ij}; \quad (19)$$

$$H(\varphi_0) = \int d^3x \mathcal{N} \mathcal{H}(\varphi_0) \quad (20)$$

is the total Hamiltonian of the local degrees of freedom,

$$\mathcal{H}(\varphi_0) = \frac{6}{\varphi_0^2} q^{ij} q^{kl} [\pi_{ik} \pi_{jl} - \pi_{ij} \pi_{kl}] + \frac{\varphi_0^2 q^{1/2}}{6} {}^{(3)}R(\bar{g}) + \mathcal{H}_f, \quad (21)$$

and

$$\mathcal{P}_i = 2[\nabla_k (q^{kl} \pi_{il}) - \nabla_i (q^{kl} \pi_{kl})] + \mathcal{P}_{if} \quad (22)$$

are the densities of energy and momentum and $\mathcal{H}_f, \mathcal{P}_f$ are contributions of the matter fields. In the following, we call the set of the field variables F (19) with the dynamic evolution parameter φ_0 the field world space. The local part of the momentum of the space metric determinant

$$\pi(t, x) := q^{ij} \pi_{ij} \quad (23)$$

is given in the class of functions with the non-zero Fourier harmonics, so that

$$\int d^3x \pi(t, x) = 0. \quad (24)$$

The geometric foundation of introducing the global variable (9) in GR was given in [24] as the assertion about the nonzero value of the second form in the whole space. This assertion (which contradicts the Dirac gauge $\pi = 0$) follows from the global energy constraint, as, in the lowest order of the Dirac perturbation theory, positive contributions of particle-like excitations to the zero Fourier harmonic of the energy constraint can be compensated only by the nonzero value of the second form.

2.3. Local constraints and equations of motion

Following Dirac [1] we formulate generalized Hamiltonian dynamics for the considered system (17). It means the inclusion of momenta for \mathcal{N} and N_i and appropriate terms with Lagrange multipliers

$$W^D = \int_{t_1}^{t_2} dt \left\{ L^D + \frac{1}{2} \partial_t (P_0 \varphi_0) \right\}, \quad L^D = L + \int d^3x (P_{\mathcal{N}} \dot{\mathcal{N}} + P_{N^i} \dot{N}^i - \lambda^0 P_{\mathcal{N}} - \lambda^i P_{N^i}). \quad (25)$$

We can define extended Dirac Hamiltonian as

$$H^D = N_0 \left[-\frac{P_0^2}{4V_0} + I^{-1}H(\varphi_0) \right] + \int d^3x (\lambda^0 P_{\mathcal{N}} + \lambda^i P_{N^i}). \quad (26)$$

The equations obtained from variation of W^D with respect to Lagrange multipliers are called first class primary constraints

$$P_{\mathcal{N}} = 0, \quad P_{N^i} = 0. \quad (27)$$

The condition of conservation of these constraints in time leads to the first class secondary constraints

$$\{H^D, P_{\mathcal{N}}\} = \mathcal{H} - \frac{\int d^3x \mathcal{N} \mathcal{H}}{V_0 \mathcal{N}^2} = 0, \quad \{H^D, P_{N^i}\} = \mathcal{P}_i = 0 \quad (28)$$

For completeness of the system we have to include set of secondary constraints. According Dirac we choose them in the form

$$\mathcal{N}(t, \vec{x}) = 1; \quad N^i(t, \vec{x}) = 0; \quad (29)$$

$$\pi(t, \vec{x}) = 0; \quad \chi^j := \partial_i (q^{-1/3} q^{ij}) = 0. \quad (30)$$

The equations of motion obtained for the considered system are

$$\frac{dF}{dT} = \frac{\partial H(\varphi_0)}{\partial P_F}, \quad -\frac{dP_F}{dT} = \frac{\partial H(\varphi_0)}{\partial F}, \quad (31)$$

where $H(\varphi_0)$ is given by the equation (20), and we introduced the invariant geometric time T

$$N_0 dt := dT. \quad (32)$$

2.4. Global constraints and equations of motion.

The physical meaning of the geometric time T , the dynamic variable φ_0 and its momentum is given by the explicit resolving of the zero-Fourier harmonic of the energy constraint

$$\frac{\delta W^E}{\delta N_0(t)} = -\frac{P_0^2}{4V_0} + H(\varphi_0) = 0. \quad (33)$$

This constraint has two solutions for the global momentum P_0 :

$$(P_0)_{\pm} = \pm 2\sqrt{V_0 H(\varphi_0)} \equiv H_{\pm}^*. \quad (34)$$

The equation of motion for this global momentum P_0 in gauge (29) takes the form

$$\frac{\delta W^E}{\delta P_0} = 0 \Rightarrow \left(\frac{d\varphi}{dT} \right)_{\pm} = \frac{(P_0)_{\pm}}{2V} = \pm \sqrt{\rho(\varphi_0)}; \quad \rho(\varphi_0) = \frac{\int d^3x \mathcal{H}}{V_0} = \frac{H(\varphi_0)}{V_0}. \quad (35)$$

The integral form of the last equation is

$$T_{\pm}(\varphi_1, \varphi_0) = \pm \int_{\varphi_1}^{\varphi_0} d\varphi \rho^{-1/2}(\varphi), \quad (36)$$

where $\varphi_1 = \varphi_0(t_1)$ is the initial data. Equation obtained by varying the action with respect to φ_0 follows independently from the set of all other constraints and equations of motion.

In quantum theory of GR (like in quantum theories of a particle), we get two Schrödinger equations

$$i\frac{d}{d\varphi_0}\Psi^\pm(F|\varphi_0, \varphi_1) = H_\pm^*(\varphi_0)\Psi^\pm(F|\varphi_0, \varphi_1) \quad (37)$$

with positive and negative eigenvalues of P_0 and normalizable wave functions with the spectral series over quantum numbers Q

$$\Psi^+(F|\varphi_0, \varphi_1) = \sum_Q A_Q^+ \langle F|Q \rangle \langle Q|\varphi_0, \varphi_1 \rangle \theta(\varphi_0 - \varphi_1) \quad (38)$$

$$\Psi^-(F|\varphi_0, \varphi_1) = \sum_Q A_Q^- \langle F|Q \rangle^* \langle Q|\varphi_0, \varphi_1 \rangle^* \theta(\varphi_1 - \varphi_0), \quad (39)$$

where $\langle F|Q \rangle$ is the eigenfunction of the reduced energy (34)

$$H_\pm^*(\varphi_0) \langle F|Q \rangle = \pm E(Q, \varphi_0) \langle F|Q \rangle \quad (40)$$

$$\langle Q|\varphi_0, \varphi_1 \rangle = \exp[-i \int_{\varphi_1}^{\varphi_0} d\varphi E(Q, \varphi)], \quad \langle Q|\varphi_0, \varphi_1 \rangle^* = \exp[i \int_{\varphi_1}^{\varphi_0} d\varphi E(Q, \varphi)]. \quad (41)$$

The coefficient A_Q^+ , in "secondary" quantization, can be treated as the operator of creation of a universe with positive energy; and the coefficient A_Q^- , as the operator of annihilation of a universe also with positive energy. The "secondary" quantization means $[A_Q^-, A_{Q'}^+] = \delta_{Q, Q'}$. The physical states of a quantum universe are formed by the action of these operators on the vacuum $\langle 0|$, $|0 \rangle$ in the form of out-state ($|Q \rangle = A_Q^+|0 \rangle$) with positive "frequencies" and in-state ($\langle Q| = \langle 0|A_Q^-$) with negative "frequencies". This treatment means that positive frequencies propagate forward ($\varphi_0 > \varphi_1$); and negative frequencies, backward ($\varphi_1 > \varphi_0$), so that the negative values of energy are excluded from the spectrum to provide the stability of the quantum system in quantum theory of GR. In other words, instead of changing the sign of energy, we change that of the dynamic evolution parameter, which leads to the causal Green function

$$G_c(F_1, \varphi_1|F_2, \varphi_2) = G_+(F_1, \varphi_1|F_2, \varphi_2)\theta(\varphi_2 - \varphi_1) + G_-(F_1, \varphi_1|F_2, \varphi_2)\theta(\varphi_1 - \varphi_2) \quad (42)$$

where $G_+(F_1, \varphi_1|F_2, \varphi_2) = G_-(F_2, \varphi_2|F_1, \varphi_1)$ is the "commutative" Green function

$$G_+(F_2, \varphi_2|F_1, \varphi_1) = \langle 0|\Psi^-(F_2|\varphi_2, \varphi_1)\Psi^+(F_1|\varphi_1, \varphi_1)|0 \rangle \quad (43)$$

For this causal convention, the geometric time (36) is always positive in accordance with the equations of motion (35)

$$\left(\frac{dT}{d\varphi_0}\right)_\pm = \pm\sqrt{\rho} \Rightarrow T_\pm(\varphi_1, \varphi_0) = \pm \int_{\varphi_1}^{\varphi_0} d\varphi \rho^{-1/2}(\varphi) \geq 0. \quad (44)$$

Thus, the causal structure of the field world space immediately leads to the arrow of the geometric time (44) and the beginning of evolution of a universe with respect to the geometric time $T = 0$.

The way to obtain conserved integrals of motion in classical theory and quantum numbers Q in quantum theory is the Levi-Civita-type canonical transformation of the field world space (F, φ_0) to a geometric set of variables (V, Q_0) with the condition that the geometric evolution parameter Q_0 coincides with the geometric time $dT = dQ_0$.

Equations (35), (36) in the homogeneous approximation of GR are the basis of observational cosmology where the geometric time is the conformal time connected with the world time T_f of the Friedmann cosmology by the relation

$$dT_f = \frac{\varphi_0(T)}{\mu} dT, \quad (45)$$

and the dependence of scale factor (dynamic evolution parameter φ_0) on the geometric time T is treated as the evolution of the universe. In particular, equation (35) gives the relation between the present-day value of the dynamic evolution parameter $\varphi_0(T_0)$ and cosmological observations, i.e., the density of matter ρ and the Hubble parameter

$$\mathcal{H}_{hub}^e = \frac{\mu\varphi_0'}{\varphi_0^2} = \frac{\mu\sqrt{\rho}}{\varphi_0^2} \Rightarrow \varphi_0(T_0) = \left(\frac{\mu\sqrt{\rho}}{\mathcal{H}_{hub}} \right)^{1/2} := \mu\Omega_0^{1/4} \quad (46)$$

where ($0.6 < (\Omega_0^{1/4})_{exp} < 1.2$). The dynamic evolution parameter as the cosmic scale factor and a value of its conjugate momentum (i.e., a value of the dynamic Hamiltonian) as the density of matter (see equations (35), (36)) are objects of measurement in observational astrophysics and cosmology and numerous discussions about the Hubble parameter, dark matter, and hidden mass.

The general theory of reparametrization-invariant reduction described in the previous Sections can be applied also to GR. In accordance with this theory, the reparametrization-invariant dynamics of GR is covered by two unconstrained systems (dynamic and geometric) connected by the Levi-Civita canonical transformation which solves the problems of the initial data, conserved quantum numbers, and direct correspondence of standard classical cosmology with quantum gravity on the level of the generating functional of the unitary and causal perturbation theory [7, 15].

3. Equivalent Unconstrained Systems

Assume that we can solve the constraint equations and pass to the reduced space of independent variables (F^*, P_F^*) . The explicit solution of the local and global constraints has two analytic branches with positive and negative values for scale factor momentum P_0 (34). Therefore, inserting solutions of all constraints into the action we get two branches of the equivalent Dynamic Unconstrained System (DUS)

$$W_{\pm}^{DUS}[F|\varphi_0] = \int_{\varphi_1}^{\varphi_2} d\varphi_0 \left\{ \left[\int d^3x \sum_{F^*} P_F^* \frac{\partial F^*}{\partial \varphi_0} \right] - H_{\pm}^* + \frac{1}{2} \partial_{\varphi_0}(\varphi_0 H_{\pm}^*) \right\}, \quad (47)$$

where φ_0 plays the role of evolution parameter and H_{\pm}^* defined by equation (34) plays the role of the evolution Hamiltonian, in the reduced phase space of independent physical variables (F^*, P_F^*) with equations of motion

$$\frac{dF^*}{d\varphi_0} = \frac{\partial H_{\pm}^*}{\partial P_F^*}, \quad -\frac{dP_F^*}{d\varphi_0} = \frac{\partial H_{\pm}^*}{\partial F^*}. \quad (48)$$

The evolution of the field world space variables (F^*, φ_0) with respect to the geometric time T is not contained in DUS (47). This geometric time evolution is described by supplementary equation (35) for nonphysical momentum P_0 (34) that follows from the initial extended system.

To get an equivalent unconstrained system in terms of the geometric time (we call it the Geometric Unconstrained System (GUS)), we need the Levi-Civita canonical transformation (LC) [12, 21] of the field world phase space

$$(F^*, P_F^*|\varphi_0, P_0) \Rightarrow (F_G^*, P_G^*|Q_0, \Pi_0) \quad (49)$$

which converts the energy constraint (33) into the new momentum Π_0 .

In terms of geometrical variables the action takes the form

$$W^G = \int_{t_1}^{t_2} dt \left\{ \left[\int d^3x \sum_{F_G^*} P_G^* \dot{F}_G^* \right] - \Pi_0 \dot{Q}_0 + N_0 \Pi_0 + \frac{d}{dt} S^{LC} \right\} \quad (50)$$

where S^{LC} is generating function of LC transformations. Then the energy constraint and the supplementary equation for the new momentum take trivial form

$$\Pi_0 = 0 ; \quad \frac{\delta W}{\delta \Pi_0} = 0 \quad \Rightarrow \quad \frac{dQ_0}{dt} = N_0 \quad \Rightarrow \quad dQ_0 = dT. \quad (51)$$

Equations of motion are also trivial

$$\frac{dP_G^*}{dT} = 0, \quad \frac{dF_G^*}{dT} = 0, \quad (52)$$

and their solutions are given by the initial data

$$P_G^* = P_G^{*0}, \quad F_G^* = F_G^{*0}. \quad (53)$$

Substituting solutions of (51) and (52) into the inverted Levi-Civita transformations

$$F^* = F^*(Q_0, \Pi_0 | F_G^*, P_G^*), \quad \varphi_0 = \varphi_0(Q_0, \Pi_0 | F_G^*, P_G^*) \quad (54)$$

and similar for momenta, we get formal solutions of (48) and (36)

$$F^* = F^*(T, 0 | F_G^{*0}, P_G^{*0}), \quad P_F^* = P_F^*(T, 0 | F_G^{*0}, P_G^{*0}), \quad \varphi_0 = \varphi_0(T, 0 | F_G^{*0}, P_G^{*0}). \quad (55)$$

We see that evolution of the dynamic variables with respect to the geometric time (i.e., the evolution of a universe) is absent in DUS. The evolution of the dynamic variables with respect to the geometric time can be described in the form of the LC (inverted) canonical transformation of GUS into DUS (54), (55).

There is also the weak form of Levi-Civita-type transformations to GUS $(F^*, P_F^*) \Rightarrow (\tilde{F}, \tilde{P})$ without action-angle variables and with a constraint

$$\tilde{\Pi}_0 - \tilde{H}(\tilde{Q}_0, \tilde{F}, \tilde{P}) = 0. \quad (56)$$

We get the constraint-shell action

$$\tilde{W}^{GUS} = \int dT \left\{ \left[\int d^3x \sum_{\tilde{F}} \tilde{P} \frac{d\tilde{F}}{dT} \right] - \tilde{H}(T, \tilde{F}, \tilde{P}) \right\}, \quad (57)$$

that allows us to choose the initial cosmological data with respect to the geometric time.

Recall that the considered reduction of the action reveals the difference of reparametrization-invariant theory from the gauge-invariant theory: in gauge-invariant theory the superfluous (longitudinal) variables are completely excluded from the reduced system; whereas, in reparametrization-invariant theory the superfluous (longitudinal) variables leave the sector of the Dirac observables (i.e., the phase space (F^*, P_F^*)) but not the sector of measurable quantities: superfluous (longitudinal) variables become the dynamic evolution parameter and dynamic Hamiltonian of the reduced theory.

4. Reparametrization-Invariant Path Integral

Following Faddeev-Popov procedure we can write down the path integral for local fields of our theory using constraints and gauge conditions (27-30):

$$Z_{\text{local}}(F_1, F_2 | P_0, \varphi_0, N_0) = \int_{F_1}^{F_2} D(F, P_f) \Delta_s \bar{\Delta}_t \exp \{i \bar{W}\}, \quad (58)$$

where

$$D(F, P_f) = \prod_{t,x} \left(\prod_{i < k} \frac{dq^{ik} d\pi_{ik}}{2\pi} \prod_f \frac{df dp_f}{2\pi} \right) \quad (59)$$

are functional differentials for the metric fields (π, q) and the matter fields (p_f, f) ,

$$\Delta_s = \prod_{t,x,i} \delta(\mathcal{P}_i) \delta(\chi^j) \det\{\mathcal{P}_i, \chi^j\}, \quad (60)$$

$$\bar{\Delta}_t = \prod_{t,x} \delta(\mathcal{H}(\mu)) \delta(\pi) \det\{\mathcal{H}(\varphi_0) - \rho, \pi\}, \quad \left(\rho = \frac{\int d^3 x H(\varphi_0)}{V_0} \right) \quad (61)$$

are the F-P determinants, and

$$\bar{W} = \int_{t_1}^{t_2} dt \left\{ \int_{V_0} d^3 x \left(\sum_F P_F \dot{F} \right) - P_0 \dot{\varphi}_0 - N_0 \left[-\frac{P_0^2}{4V_0} + H(\varphi_0) \right] + \frac{1}{2} \partial_t (P_0 \varphi_0) \right\} \quad (62)$$

is extended action of considered theory.

By analogy with a particle and a string considered in papers [15, 16] we define a commutative Green function as an integral over global fields (P_0, φ_0) and the average over reparametrization group parameter N_0

$$G_+(F_1, \varphi_1 | F_2, \varphi_2) = \int_{\varphi_1}^{\varphi_2} \prod_t \left(\frac{d\varphi_0 dP_0 d\tilde{N}_0}{2\pi} \right) Z_{\text{local}}(F_1, F_2 | P_0, \varphi_0, N_0), \quad (63)$$

where

$$\tilde{N} = N/2\pi \delta(0), \quad \delta(0) = \int dN_0. \quad (64)$$

The causal Green function in the world field space (F, φ_0) is defined as the sum

$$G_c(F_1, \varphi_1 | F_2, \varphi_2) = G_+(F_1, \varphi_1 | F_2, \varphi_2) \theta(\varphi_1 - \varphi_2) + G_+(F_2, \varphi_1 | F_2, \varphi_1) \theta(\varphi_2 - \varphi_1). \quad (65)$$

This function will be considered as generating functional for the unitary S -matrix elements [26]

$$S[1, 2] = \langle \text{out } (\varphi_2) | T_\varphi \exp \left\{ -i \int_{\varphi_1}^{\varphi_2} d\varphi (H_I^*) \right\} | (\varphi_1) \text{ in } \rangle, \quad (66)$$

where T_φ is a symbol of ordering with respect to parameter φ_0 , and $\langle \text{out } (\varphi_2) |$, $| (\varphi_1) \text{ in } \rangle$ are states of quantum Universe in the lowest order of the Dirac perturbation theory ($\mathcal{N} = 1$; $N^k = 0$; $q^{ij} = \delta_{ij} + h_{ij}^T$), H_I^* is the interaction Hamiltonian

$$H_I^* = H^* - H_0^*, \quad H^* = 2\sqrt{V_0 H(\varphi)}, \quad H_0^* = 2\sqrt{V_0 H_0(\varphi)}, \quad (67)$$

H_0 is a sum of the Hamiltonians of "free" fields (gravitons, photons, massive vectors, and spinors) where all masses (including the Planck mass) are replaced by the dynamic evolution parameter φ_0 [7]. For example for gravitons the "free" Hamiltonian takes the form:

$$H_0(\varphi_0) = \int d^3x \left(\frac{6(\pi_{(h)}^T)^2}{\varphi_0^2} + \frac{\varphi_0^2}{24} (\partial_i h^T)^2 \right); \quad (h_{ii}^T = 0; \quad \partial_j h_{ji}^T = 0). \quad (68)$$

4.1. QFT limit of Quantum Gravity

The simplest way to determine the QFT limit of Quantum Gravity and to find the region of validity of the FP-integral is to use the quantum field version of the reparametrization-invariant integral (63) in the form of S-matrix elements [26] (see (66), (67)). We consider the infinite volume limit of the S-matrix element (67) in terms of the geometric time T for the present-day stage $T = T_0, \varphi(T_0) = \mu$, and $T(\varphi_1) = T_0 - \Delta T, T(\varphi_2) = T_0 + \Delta T = T_{out}$. One can express this matrix element in terms of the time measured by an observer of an out-state with a tremendous number of particles in a universe using equation $d\varphi = dT_{out} \sqrt{\rho_{out}}$ and approximation of a tremendous energy ($10^{79} GeV$) in comparison with possible real and virtual deviations of the free Hamiltonian in the laboratory processes:

$$\bar{H}_0 = E_{out} + \delta H_0, \quad \langle out | \delta H_0 | in \rangle \ll E_{out}. \quad (69)$$

to neglect "back-reaction".

In the infinite volume limit, we get from (67)

$$d\varphi_0[H_I^*] = 2d\varphi_0 \left(\sqrt{V_0(H_0 + H_I)} - \sqrt{V_0 H_0} \right) = dT_{out} [\hat{F} \bar{H}_I + O(1/E_{out})] \quad (70)$$

where H_I is the interaction Hamiltonian in GR, and

$$\hat{F} = \sqrt{\frac{E_{out}}{H_0}} = \sqrt{\frac{E_{out}}{E_{out} + \delta H_0}} \quad (71)$$

is a multiplier which plays the role of a form factor for physical processes observed in the "laboratory" conditions when the cosmic energy E_{out} is much greater than the deviation of the free energy

$$\delta H_0 = H_0 - E_{out}; \quad (72)$$

due to creation and annihilation of real and virtual particles in the laboratory experiments.

The measurable time of the laboratory experiments $T_2 - T_1$ is much smaller than the age of the universe T_0 , but it is much greater than the reverse "laboratory" energy δ , so that the limit

$$\int_{T(\varphi_1)}^{T(\varphi_2)} dT_{out} \Rightarrow \int_{-\infty}^{+\infty} dT_{out}$$

is valid. If we neglect the form factor (71) that removes a set of ultraviolet divergences, we get the standard S-matrix element [19] that corresponds to the standard FP functional integral with the geometric (conformal) time T (instead of the coordinate time t) and with conformal-invariant fields $t \rightarrow T_{out}$:

$$S[-\infty | +\infty] = \langle out | T \exp \left\{ -i \int_{-\infty}^{+\infty} dT_{out} \hat{F} H_I(\mu) \right\} | in \rangle \quad (\hat{F} = 1). \quad (73)$$

Thus, the standard FP-integral and the unitary S-matrix for conventional quantum field theory (QFT) appears as the nonrelativistic approximation of tremendous mass of a universe and its very large life-time. Now, it is evident that QFT are not valid for the description of the early universe given in the finite spatial volume and the finite positive interval of geometrical time ($0 \leq T \leq T_0$) where T_0 is the "present-day value" for the early universe that only begins to create matter.

On the other hand, we revealed that standard QFT (that appears as the limit of quantum theory of the Einstein general relativity) speaks on the language of the conformal fields and coordinates. If we shall consider the standard QFT as the limit case of quantum gravity, we should recognize that, in QFT, we measure the conformal quantities, as QFT is expressed in terms of the conformal-invariant Lichnerowicz variables and coordinates including the conformal time (T_{out}) as the time of evolution of these variables.

The conformal invariance of the variables can testify to the conformal invariance of the initial theory of gravity of the type of a dilaton version of GR given in a space with the geometry of similarity [4, 5].

5. Conclusions

All relativistic theories, including general relativity considered in the present paper, are given in their world spaces of dynamic variables by their singular actions (as integrals over the coordinate space) and by the geometric interval.

The peculiarity of general relativity is the invariance of its action and the geometric interval with respect to reparametrization of the coordinate space, i.e., the general coordinate transformations.

The main peculiarity of general relativity (which we tried to reveal in the paper) is the following: the reparametrization symmetry means that the measurable geometric time is a time-like variable in the field world space (obtained by the Levi-Civita transformation to the action-angle-type variables) rather than the coordinate.

The dynamic origin of the "time" was reliably covered by the gauge condition that the lapse-function is equal to unity.

This non-invariant gauge-fixing method of describing the Hamiltonian dynamics of general relativity was a real obstacle for understanding this dynamics. This non-invariant method confuses reparametrization-invariant (or measurable) quantities and non-invariant (nonobservable) ones and hides the necessity of constraining by the Levi-Civita transformation that converts ambiguous and attractive "mathematical games" with non-invariant quantities into a harmonious theory of invariant dynamics in the world space which includes an unambiguous description of quantum gravity with its relation to the standard cosmology of a classical universe.

The generating functionals for causal Green functions of the unitary perturbation theory in the form of path integrals are constructed by averaging over a space of the reparametrization group, instead of the gauge-fixing.

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